

Memory difference control of unknown unstable fixed points: Drifting parameter conditions and delayed measurement

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Difference control schemes for controlling unstable fixed points become important if the exact position of the fixed point is unavailable or moving due to drifting parameters. We propose a memory difference control method for stabilization of *a priori* unknown unstable fixed points by introducing a memory term. If the amplitude of the control applied in the previous time step is added to the present control signal, fixed points with arbitrary Lyapunov numbers can be controlled. This method is also extended to compensate arbitrary time steps of measurement delay. We show that our method stabilizes orbits of the Chua circuit where ordinary difference control fails. [S1063-651X(98)09712-8]

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I. INTRODUCTION

Two main classes of control strategies are established in chaos control: The Ott-Grebogi-Yorke (OGY) control algorithm [1], almost a standard method for controlling chaos, does not provide any readjustment of the fixed point during *in situ* measurements without loss of control. In many experimental systems, however, it is desirable to use a control strategy that does not rely on the knowledge of the exact position of the fixed point, because the location of the fixed points can change due to drifts in parameters. On the other hand, the time-continuous control method proposed by Pyragas [2] is practically limited by the required sampling rate, and does not allow stabilization of arbitrary orbits as has recently been shown in [3].

Both problems are circumvented by simple time-discrete difference control [4]. It is limited to a certain range of Lyapunov numbers. Control of arbitrary periodic orbits can be achieved if the algorithm is applied only every second period [4,5] or by a memory method.

In this paper we present an improved memory difference control (MDC) method that takes control amplitudes into account that were applied at previous time steps. MDC allows one to stabilize drifting fixed points with arbitrary Lyapunov numbers and shows an enlarged region of stability.

This method is generalized when dealing with measurements delayed by τ time steps (orbit periods). This task is accomplished by increasing the number of memorized control amplitudes by τ . Given the stable and unstable directions of the fixed point with sufficient accuracy, only one accessible control parameter for each unstable direction is required to achieve control.

We compare difference control and MDC at the well-known Chua oscillator [6] and show that orbits for which difference control fails are stabilized by MDC.

II. STABILIZATION OF FIXED POINTS BY DIFFERENCE CONTROL

In experimental situations a Poincaré section is commonly used to reduce the dynamics to a time-discrete description by an iterated map

$$\vec{x}_{t+1} = \vec{f}(\vec{x}_t, \vec{r}). \quad (1)$$

Here \vec{r} denotes a set of control parameters that are in the unperturbed dynamics assumed to be constant or varying on a slow time scale compared to the length of a period orbit.

The idea to control chaos by small perturbations of control parameters implies that \vec{r} becomes time-dependent. The time-dependent control parameter \vec{r}_t is updated at each discrete time step defined by the Poincaré section. Its value is determined according to the specific control algorithm and is kept constant for a part of the orbit. Without loss of generality we choose $\vec{x}^* = \vec{0}$ and $\vec{r} = \vec{0}$ for the fixed point to be stabilized. The linearized equation of motion around the unstable fixed point then becomes

$$\vec{x}_{t+1} = L\vec{x}_t + M\vec{r}_t, \quad (2)$$

where

$$L_{ij} := \left. \frac{\partial f_i}{\partial x_j} \right|_{\vec{x}^* = \vec{0}, \vec{r} = \vec{0}} \quad \text{and} \quad M_{ij} := \left. \frac{\partial f_i}{\partial r_j} \right|_{\vec{x}^* = \vec{0}, \vec{r} = \vec{0}}. \quad (3)$$

The Lyapunov numbers of the orbit are the eigenvalues of L . Here one has to distinguish the Lyapunov number of an orbit (or time-discrete map) from the local (or conditional) Lyapunov exponent and the commonly used global Lyapunov exponent being an ergodic average over the attractor [7]. In principle it is possible to proceed with a multiparameter control by using as many control parameters as there are degrees of freedom, i.e., $\dim(\vec{r}) = \dim(\vec{x})$. Instead, it is common to follow Ott, Grebogi, and Yorke [1] to transform the system to the eigensystem of L . Control is applied *only* in the unstable subspace [8]. The evolution of the equation of motion is again of the form of Eq. (2) with reduced dimension of L .

In difference control [4] the control parameter is updated at the Poincaré section according to

$$\vec{r}_t = K(\vec{x}_t - \vec{x}_{t-1}). \quad (4)$$

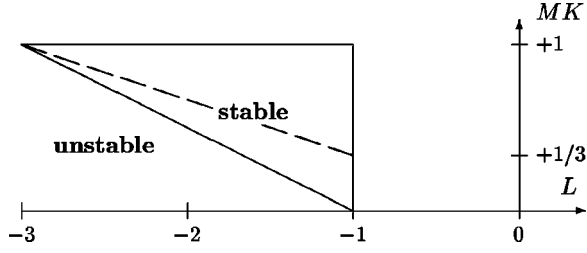


FIG. 1. Stability area for difference control (one unstable Lyapunov number L): For $|L| < 1$ the system is stable without control. Fixed points with Lyapunov number $-3 < L < -1$ can be controlled if MK is chosen within the area bounded by the triangle. The line within the triangle shows the optimal value for MK .

In contrast to OGY control, difference control is limited to fixed points with Lyapunov numbers between -3 and -1 [4,5]. A stability diagram [9,5] for the case of one unstable eigenvalue is shown in Fig. 1.

Simple difference control uses information that is partially out of date, resulting in an additional degree of freedom from the delayed amplitude \vec{x}_{t+1} . This fact is illustrated by imagining two agents controlling the same system in turns. If they do not communicate, control fails because of the inherent delay of the system to be controlled. This effect is compensated when using the information contained in the previous value of the control amplitude \vec{r}_{t-1} .

III. MEMORY DIFFERENCE CONTROL

We define memory difference control by [10]

$$\vec{r}_t = K(\vec{x}_t - \vec{x}_{t-1}) + N\vec{r}_{t-1}. \quad (5)$$

Combined with Eq. (2), we obtain a dynamical system which reads in delayed coordinates for \vec{x} and \vec{r} ,

$$\begin{pmatrix} \vec{x}_{t+1} \\ \vec{x}_t \\ \vec{r}_t \end{pmatrix} = \begin{pmatrix} L + MK & -MK & MN \\ 1 & 0 & 0 \\ K & -K & N \end{pmatrix} \begin{pmatrix} \vec{x}_t \\ \vec{x}_{t-1} \\ \vec{r}_{t-1} \end{pmatrix}. \quad (6)$$

In order to stabilize the fixed point, all eigenvalues α_i of the matrix in Eq. (6) must have a modulus smaller than 1. This ensures exponential convergence to the fixed point. If all parameter values are chosen such that all eigenvalues become zero, the fixed point is reached after a finite number of time steps. In fact this can be guaranteed by MDC. We first assume that M and $(L-1)$ are both invertible, and that the number of accessible control parameters is equal to or greater than the dimension of the unstable manifold, i.e., $\dim(\vec{r}) \geq \dim(\vec{x})$. Then all eigenvalues are zero [11,12] if

$$\begin{aligned} K &= -M^{-1}L^2(L-1)^{-1}, \\ N &= M^{-1}L(L-1)^{-1}M. \end{aligned} \quad (7)$$

The concept of MDC can be generalized to stabilization of (known and unknown) fixed points when \vec{x} can be measured only after a finite number of delay steps [11,13]: If the system is measured with τ steps delay, Eq. (7) is replaced by

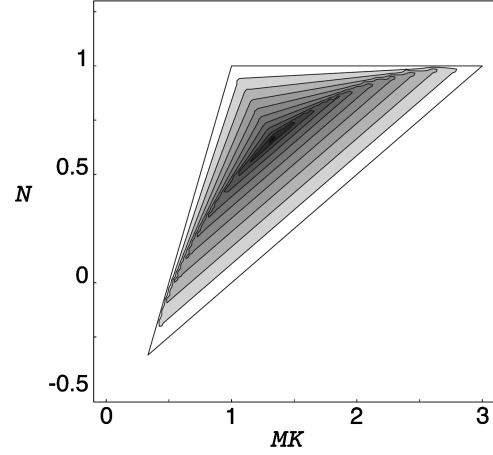


FIG. 2. Stability region of memory difference control for $L = -2$ in the (MK, N) plane. Within the triangle MDC is stable; the contour lines show the decrease of the largest eigenvalue to zero in $(MK, N) = (4/3, 2/3)$.

$$K = -M^{-1}L^{\tau+2}(L-1)^{-1}, \quad (8)$$

$$N_i = -M^{-1}L^i M, \quad \forall 1 \leq i \leq \tau, \quad (9)$$

$$N_{\tau+1} = M^{-1}L^{\tau+1}(L-1)^{-1}M, \quad (10)$$

where the feedback now contains a sum over $\tau+1$ preceding control amplitudes:

$$\vec{r}_t = K(\vec{x}_{t-\tau} - \vec{x}_{t-\tau-1}) + \sum_{i=1}^{\tau+1} N_i \vec{x}_{t-\tau-i}. \quad (11)$$

A similar control scheme can be applied for OGY control by choosing $K = -M^{-1}L^{\tau+1}$ and $N_i = -M^{-1}L^i M$ ($1 \leq i \leq \tau$). For details see [11,13].

IV. THE STABILITY DIAGRAM IN THE ONE-DIMENSIONAL CASE

Since the optimal control values are never exactly matched in experiments, it is important to know the complete stability diagram in (K, N) , in particular the optimal value of K for given N and vice versa. In the one-dimensional case, the characteristic equation is given by

$$0 = \alpha[\alpha^2 - (L + MK + N)\alpha + (MK + NL)]. \quad (12)$$

The stability region in the (K, N) plane, i.e., where all eigenvalues have modulus smaller than 1, is the triangle shown in Fig. 2. Its corners are given by

$$(MK, N)_{+1, +1} = (1 - L, 1), \quad (13)$$

$$(MK, N)_{-1, -1} = \left(-\frac{(L+1)^2}{(L-1)}, \frac{(L+3)}{(L-1)} \right), \quad (14)$$

$$(MK, N)_{+1, -1} = (-1 - L, 1), \quad (15)$$

where the eigenvalues take the values $+1$ and -1 as indicated by the indices. Two sides of the triangle are determined by the conditions that one eigenvalue is equal to $+1$ and

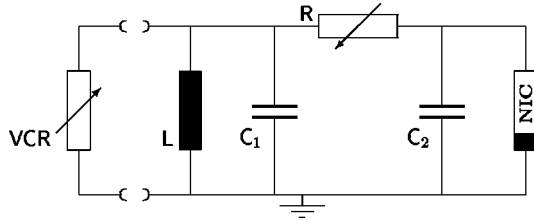


FIG. 3. Chua circuit: The negative resistance (NIC) is both non-linearity and energy source of the circuit. Rough adjustment of the control parameter can be done by adjusting R . Control is applied with the voltage-dependent resistor (VCR).

-1 , respectively. The third is given by $MK + NL = +1$ where the eigenvalues are a complex conjugate pair on the unit circle. The line that determines N with minimal eigenvalues for a given K (and vice versa) is given by the algebraic expression

$$0 = (MK)^2 + 2MKN + N^2 + (2L - 4)MK - 2LN + L^2,$$

where both eigenvalues coincide.

V. THE CHUA OSCILLATOR

We demonstrate the efficiency of the improved difference method by controlling unstable periodic orbits of the chaotic attractor of Chua's oscillator [6]. The Chua circuit is an autonomous system (see Fig. 3). The Poincaré section necessary for control is obtained by an electronic zero-crossing detector. The frequency is in the range of $\nu = 2\pi/\sqrt{LC_1} \sim 3.6$ kHz and allows the usage of digital signal processing tools to implement control algorithms.

Parameter drifts, e.g., temperature drifts, naturally occur in electronic circuits and difference control methods have the advantage that they follow the drifting fixed point. In order to get access to an appropriate control parameter, a VCR (voltage controlled resistor) has been added to the circuit. The basic dynamics is nevertheless determined by the resistor R .

Furthermore, the Chua circuit allows us to investigate interesting ranges of Lyapunov numbers simply by choosing different values of the main control parameter R .

The dynamics of the Chua circuit is described, in first

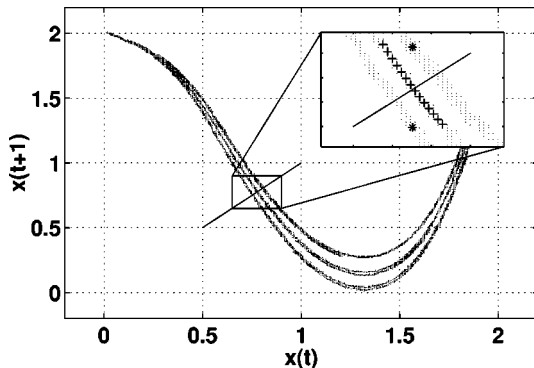


FIG. 4. Return map $x_{t+1}(x_t)$ in the Poincaré surface of section for three different values of the control parameter. From these data one obtains values of L and M to adjust the optimal parameters of control.

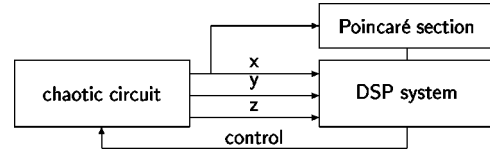


FIG. 5. Implementation of the control method: One coordinate is used to generate the Poincaré section. A digital signal processor is used for measurement, computation, and application of the control amplitude.

approximation, by the normalized equations

$$\begin{aligned} \dot{u} &= \alpha[v - u - f(u)], \\ \dot{v} &= u - v + w, \\ \dot{w} &= -\beta v, \end{aligned} \quad (16)$$

where f is the input-output function of the negative resistance approximately described by the piecewise linear descending function

$$f(u) = m_0 u + \frac{1}{2}(m_1 - m_0)(|u + 1| - |u - 1|), \quad (17)$$

with $m_0 > m_1$ [14]. Rather than solving these equations numerically, we demonstrate the stabilization of an unstable periodic orbit directly in the electronic system.

VI. IMPROVED DIFFERENCE CONTROL OF THE CHUA SYSTEM

The standard control strategy is to measure the required system variables, generate the Poincaré map for, e.g., three adjacent values of the control parameter (Fig. 4), and to calculate the parameters of the feedback to the control parameter (Fig. 5).

In the present case the return map $x_{t+1}(x_t, z_t)$ is approximately a function of x_t alone. Therefore only two variables are required. The first one, y_t , is used to determine the Poincaré surface of section by a zero crossing detector. The second one, x_t , is used for the calculation of the control.

The stability region for different values of the memory

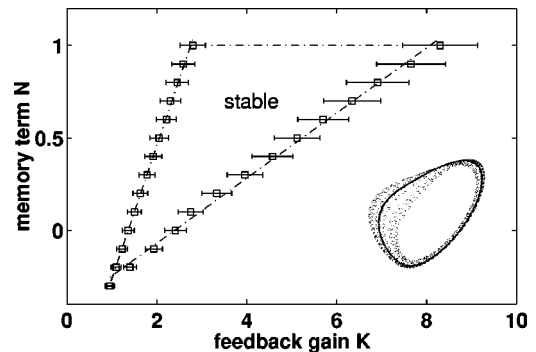


FIG. 6. Stabilization of an orbit in the single-scroll chaotic attractor. Within the measured triangle memory difference control stabilizes the orbit of the Chua circuit. The special case of simple difference control is given by $N=0$. The inset shows the attractor and the stabilized periodic orbit.

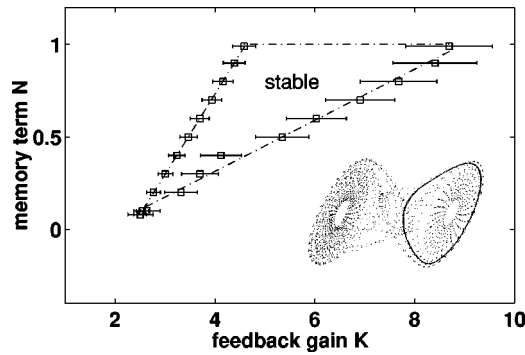


FIG. 7. Stabilization of an orbit in the double-scroll chaotic attractor: In agreement with theory, control without memory ($N=0$) fails, due to a Lyapunov number $L < -3$. Within the triangle the orbit is stabilized by memory difference control. Again, attractor and orbit are shown in the inset.

term N and feedback gain K is measured by changing the values until control is entirely lost. The lower bound of K is easily recognized by a doubling of the stabilized period. However, the upper bound of K , where the loss of control is noise-induced, is more difficult to estimate. In Fig. 6 the stability region for a stabilized orbit in the single-scroll chaos is shown. The corresponding Lyapunov number $L = -2.15 \pm 0.04$ has been calculated from the Poincaré map. The stability diagram includes the stability region of simple difference control as the special case ($N=0$). Stabilization of the same periodic orbit in the double-scroll chaotic attractor is not possible with simple difference control. This is due to a Lyapunov number of $L = -3.27 \pm 0.08$ for which the method is predicted to fail.

The stability region of the stabilized orbit in the single-scroll (Fig. 6) and double-scroll attractor (Fig. 7) have a broad overlap. Thus it is possible to choose universal values of (K, N) that allow tracking of an orbit from one regime to the other without changing parameters of the controlling circuit.

The significant improvement of memory difference control compared to simple difference control is demonstrated by the estimation of Lyapunov exponents (contraction rates) from the transient. Figure 8 shows the stability regimes for different feedback gains K . Simple difference control corresponds to $N=0$, and MDC to $N=0.7$. The range of controllability is broadened and the Lyapunov exponents are smaller, equivalent to faster convergence. The measurements are in good agreement with our theoretical predictions (Fig. 8). Due to noise-induced loss of control, it was impossible to obtain reliable measurements for large K of the case $N=0.7$.

VII. REESTIMATION OF LYAPUNOV NUMBERS FROM THE BORDERS OF STABILITY

From the borders and corners of the measured stability region, the exact values of the Lyapunov number of the con-

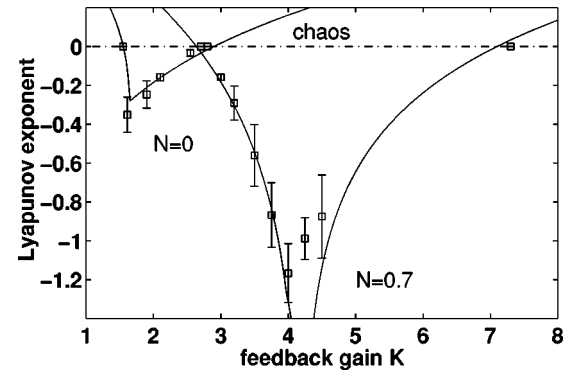


FIG. 8. Measured Lyapunov exponents of the control transients for $N=0$ and $N=0.7$ (single-scroll regime) compared to theoretical Lyapunov exponents (lines) from the eigenvalue analysis.

trolled cycle are reestimated, similar to the approach used in [15] for OGY control of the Hénon map. Since two values of N are exactly 1, four of the six coordinate values can be used to determine L and M by a least-square fit weighted by the variances of the measured values giving

$$L_{ss} = -2.069 \pm 0.03, \quad L_{ds} = -3.24 \pm 0.03, \quad (18)$$

$$M_{ss} = 0.376 \pm 0.015, \quad M_{ds} = 0.488 \pm 0.005,$$

for the orbit stabilized in the single-scroll (ss) and double-scroll (ds) attractor, respectively. These values are in good agreement with the values given in Sec. VI, which were obtained from the Poincaré map (Fig. 4).

VIII. CONCLUSIONS

The dynamical behavior and stability conditions of difference control and memory difference control are fundamentally different from the stability conditions [3] of time-continuous Pyragas control [2]. In this paper we introduced and discussed memory difference control as a powerful method to stabilize unstable fixed points even in the presence of parameter drift or delayed measurement. Investigations of the Chua oscillator circuit demonstrated the reliability of the method.

Memory difference control overcomes the Lyapunov number limitations of difference control and thus appears to be superior both to OGY and Pyragas control schemes.

ACKNOWLEDGMENTS

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- [8] Once the stable and unstable directions are known, the Poincaré iteration dynamics reduces to the motion in unstable subspace. Note that the unstable subspace needs only as many dimensions as the system has unstable eigenvalues. In the simplest case, the Lyapunov exponents are of signature $(+, 0, -)$ and the unstable subspace of dimension 1.
- [9] This follows from the eigenvalues in Eq (6) for $N=0$.
- [10] One could think of extending difference control by using delayed differences $x_{t-i} - x_{t-i-1}$ of the system variable instead of previous control values r_{t-i} . This is an approach that improves OGY-fashioned control if the system is measured with *only one step* delay [11]. Unfortunately, for difference control one does not obtain a stable control scheme from this approach.
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